More Bayes, Law of Total Probability, and Independence

Practice problems at the end

```r
df <- function(n){
  S <- sample(c("setosa","versicolor", "virginica"), n, replace=TRUE)
  pc <- .4*(S=="setosa") + .5*(S=="versicolor") + .2
  C <- c("purple","pink")[rbinom(n,1,pc)+1]
  data.frame(S = S, C = C)
}
```

	Species			
Color	Setosa	Versicolor	Virginica	Row Total
pink				
Cell prob	?	?	?	?
Row prob	?	?	?	
Col prob	?	?	?	
purple				
Cell prob	?	?	?	?
Row prob	?	?	?	
Col prob	?	?	?	
Column Total	?	?	?	?

```r
df <- function(n){
  S <- sample(c("setosa","versicolor", "virginica"), n, replace=TRUE)
  pc <- .4*(S=="setosa") + .5*(S=="versicolor") + .2
  C <- c("purple","pink")[rbinom(n,1,pc)+1]
  data.frame(S = S, C = C)
}
```

	Species			
Color	Setosa	Versicolor	Virginica	Row Total
pink				
Cell prob				
Row prob				
Col prob				
purple				
Cell prob				
Row prob				
Col prob				
Column Total				

Compare to simulation results

```
df1 <- df(1000000)
gmodels::CrossTable(df1$C,df1$S)
```

From last time …

pet		blue	green	red	Row Total
cat					
	Cell prob	?	?	?	0.3
	Row prob	0.2	0.5	?	
	Col prob	?	?	?	
dog					
	Cell prob	?	?	?	?
	Row prob	0.3	?	0.6	
	Col prob	?	?	?	
Column Total		?	?	?	?

Question: Is there enough information to fill in the rest of the table?

pet	blue	green	red	Row Total
cat				
Cell prob	?	?	?	0.3
Row prob	0.2	0.5	0.3	
Col prob	?	?	?	
dog				
Cell prob	?	?	?	0.7
Row prob	0.3	0.1	0.6	
Col prob	?	?	?	
Column Total	?	?	?	?

pet	blue	green	red	Row Total
cat				
Cell prob	0.3×0.2	0.3×0.5	0.3×0.3	0.3
Row prob	0.2	0.5	0.3	
Col prob	?	?	?	
dog				
Cell prob	?	?	?	0.7
Row prob	0.3	0.1	0.6	
Col prob	?	?	?	
Column Total	?	?	?	?

pet	blue	green	red	Row Total
cat				
Cell prob	0.3×0.2	0.3×0.5	0.3×0.3	0.3
Row prob	0.2	0.5	0.3	
Col prob	?	?	?	
dog				
Cell prob	0.7×0.3	0.7×0.1	0.7×0.6	0.7
Row prob	0.3	0.1	0.6	
Col prob	?	?	?	
Column Total	?	?	?	?

pet	blue	green	red	Row Total
cat				
Cell prob	0.3×0.2	0.3×0.5	0.3×0.3	0.3
Row prob	0.2	0.5	0.3	
Col prob	?	?	?	
dog				
Cell prob	0.7×0.3	0.7×0.1	0.7×0.6	0.7
Row prob	0.3	0.1	0.6	
Col prob	?	?	?	
Column Total	$0.3 \times 0.2 + 0.7 \times 0.3$	$0.3 \times 0.5 + 0.7 \times 0.1$	$0.3 \times 0.3 + 0.7 \times 0.6$	1

pet	blue	green	red	Row Total
cat Cell prob Row prob Col prob	 0.3×0.2 0.2 $\frac{0.3\times0.2}{0.3\times0.2+0.7\times0.3}$	 0.3×0.5 0.5 $\frac{0.3\times0.5}{0.3\times0.5+0.7\times0.1}$	 0.3×0.3 0.3 $\frac{0.3\times0.3}{0.3\times0.3+0.7\times0.6}$	 0.3
dog Cell prob Row prob Col prob	 0.7×0.3 0.3 $\frac{0.7\times0.3}{0.3\times0.2+0.7\times0.3}$	 0.7×0.1 0.1 $\frac{0.7\times0.1}{0.3\times0.5+0.7\times0.1}$	 0.7×0.6 0.6 $\frac{0.7\times0.6}{0.3\times0.3+0.7\times0.6}$	 0.7
Column Total	$0.3 \times 0.2 + 0.7 \times 0.3$	$0.3 \times 0.5 + 0.7 \times 0.1$	$0.3 \times 0.3 + 0.7 \times 0.6$	1

Let’s repeat the calculations, but this time let’s use symbolic placeholders ….

This is the information we started with

pet	blue	green	red	Row Total
cat Cell prob Row prob Col prob	 $P(\text{blue}\mid\text{cat})$	 $P(\text{blue}\mid\text{cat})$	 $P(\text{red}\mid\text{cat})$	 $P(\text{cat})$
dog Cell prob Row prob Col prob	 $P(\text{blue}\mid\text{dog})$	 $P(\text{green}\mid\text{dog})$	 $P(\text{red}\mid\text{dog})$	 $P(\text{dog})$
Column Total				1

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat})P(\text{blue}\|\text{cat})$	$P(\text{cat})P(\text{green}\|\text{cat})$	$P(\text{cat})P(\text{red}\|\text{cat})$	$P(\text{cat})$
Row prob	$P(\text{blue}\|\text{cat})$	$P(\text{blue}\|\text{cat})$	$P(\text{red}\|\text{cat})$	
Col prob	$\frac{P(\text{cat})P(\text{blue}\|\text{cat})}{P(\text{blue})}$	$\frac{P(\text{cat})P(\text{green}\|\text{cat})}{P(\text{green})}$	$\frac{P(\text{cat})P(\text{red}\|\text{cat})}{P(\text{red})}$	
dog				
Cell prob	$P(\text{dog})P(\text{blue}\|\text{dog})$	$P(\text{dog})P(\text{green}\|\text{dog})$	$P(\text{dog})P(\text{red}\|\text{dog})$	$P(\text{dog})$
Row prob	$P(\text{blue}\|\text{dog})$	$P(\text{green}\|\text{dog})$	$P(\text{red}\|\text{dog})$	
Col prob	$\frac{P(\text{dog})P(\text{blue}\|\text{dog})}{P(\text{blue})}$	$\frac{P(\text{dog})P(\text{green}\|\text{dog})}{P(\text{green})}$	$\frac{P(\text{dog})P(\text{red}\|\text{dog})}{P(\text{red})}$	
Column Total	$P(\text{blue}) = P(\text{cat})P(\text{blue}\|\text{cat}) + P(\text{dog})P(\text{blue}\|\text{dog})$	$P(\text{green}) = P(\text{cat})P(\text{green}\|\text{cat}) + P(\text{dog})P(\text{green}\|\text{dog})$	$P(\text{red}) = P(\text{cat})P(\text{red}\|\text{cat}) + P(\text{dog})P(\text{red}\|\text{dog})$	1

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat})P(\text{blue}\|\text{cat})$	$P(\text{cat})P(\text{green}\|\text{cat})$	$P(\text{cat})P(\text{red}\|\text{cat})$	$P(\text{cat})$
Row prob	$P(\text{blue}\|\text{cat})$	$P(\text{blue}\|\text{cat})$	$P(\text{red}\|\text{cat})$	
Col prob				
dog				
Cell prob	$P(\text{dog})P(\text{blue}\|\text{dog})$	$P(\text{dog})P(\text{green}\|\text{dog})$	$P(\text{dog})P(\text{red}\|\text{dog})$	$P(\text{dog})$
Row prob	$P(\text{blue}\|\text{dog})$	$P(\text{green}\|\text{dog})$	$P(\text{red}\|\text{dog})$	
Col prob				
Column Total				1

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat})P(\text{blue}\mid\text{cat})$	$P(\text{cat})P(\text{green}\mid\text{cat})$	$P(\text{cat})P(\text{red}\mid\text{cat})$	$P(\text{cat})$
Row prob	$P(\text{blue}\mid\text{cat})$	$P(\text{blue}\mid\text{cat})$	$P(\text{red}\mid\text{cat})$	
Col prob	$\frac{P(\text{cat})P(\text{blue}\mid\text{cat})}{P(\text{blue})}$	$\frac{P(\text{cat})P(\text{green}\mid\text{cat})}{P(\text{green})}$	$\frac{P(\text{cat})P(\text{red}\mid\text{cat})}{P(\text{red})}$	
dog				
Cell prob	$P(\text{dog})P(\text{blue}\mid\text{dog})$	$P(\text{dog})P(\text{green}\mid\text{dog})$	$P(\text{dog})P(\text{red}\mid\text{dog})$	$P(\text{dog})$
Row prob	$P(\text{blue}\mid\text{dog})$	$P(\text{green}\mid\text{dog})$	$P(\text{red}\mid\text{dog})$	
Col prob	$\frac{P(\text{dog})P(\text{blue}\mid\text{dog})}{P(\text{blue})}$	$\frac{P(\text{dog})P(\text{green}\mid\text{dog})}{P(\text{green})}$	$\frac{P(\text{dog})P(\text{red}\mid\text{dog})}{P(\text{red})}$	
Column Total	$P(\text{blue}) = P(\text{cat})P(\text{blue}\mid\text{cat}) + P(\text{dog})P(\text{blue}\mid\text{dog})$	$P(\text{green}) = P(\text{cat})P(\text{green}\mid\text{cat}) + P(\text{dog})P(\text{green}\mid\text{dog})$	$P(\text{red}) = P(\text{cat})P(\text{red}\mid\text{cat}) + P(\text{dog})P(\text{red}\mid\text{dog})$	1

pet	blue	green	red	Row Total
cat Cell prob Row prob Col prob	$P(\text{cat})P(\text{blue}\|\text{cat})$ $P(\text{blue}\|\text{cat})$ $\frac{P(\text{cat})P(\text{blue}\|\text{cat})}{P(\text{blue})}$	$P(\text{cat})P(\text{green}\|\text{cat})$ $P(\text{blue}\|\text{cat})$	$P(\text{cat})P(\text{red}\|\text{cat})$ $P(\text{red}\|\text{cat})$	$P(\text{cat})$
dog Cell prob Row prob Col prob	$P(\text{dog})P(\text{blue}\|\text{dog})$ $P(\text{blue}\|\text{dog})$ $\frac{P(\text{dog})P(\text{blue}\|\text{dog})}{P(\text{blue})}$	$P(\text{dog})P(\text{green}\|\text{dog})$ $P(\text{green}\|\text{dog})$	$P(\text{dog})P(\text{red}\|\text{dog})$ $P(\text{red}\|\text{dog})$	$P(\text{dog})$
Column Total	$P(\text{blue}) =$ $P(\text{cat})P(\text{blue}\|\text{cat})+$ $P(\text{dog})P(\text{blue}\|\text{dog})$	$P(\text{green}) =$ $P(\text{cat})P(\text{green}\|\text{cat})+$ $P(\text{dog})P(\text{green}\|\text{dog})$	$P(\text{red}) =$ $P(\text{cat})P(\text{red}\|\text{cat})+$ $P(\text{dog})P(\text{red}\|\text{dog})$	1

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat})P(\text{blue}\vert\text{cat})$	$P(\text{cat})P(\text{green}\vert\text{cat})$	$P(\text{cat})P(\text{red}\vert\text{cat})$	$P(\text{cat})$
Row prob	$P(\text{blue}\vert\text{cat})$	$P(\text{blue}\vert\text{cat})$	$P(\text{red}\vert\text{cat})$	
Col prob	$\frac{P(\text{cat})P(\text{blue}\vert\text{cat})}{P(\text{blue})}$	$\frac{P(\text{cat})P(\text{green}\vert\text{cat})}{P(\text{green})}$	$\frac{P(\text{cat})P(\text{red}\vert\text{cat})}{P(\text{red})}$	
dog				
Cell prob	$P(\text{dog})P(\text{blue}\vert\text{dog})$	$P(\text{dog})P(\text{green}\vert\text{dog})$	$P(\text{dog})P(\text{red}\vert\text{dog})$	$P(\text{dog})$
Row prob	$P(\text{blue}\vert\text{dog})$	$P(\text{green}\vert\text{dog})$	$P(\text{red}\vert\text{dog})$	
Col prob	$\frac{P(\text{dog})P(\text{blue}\vert\text{dog})}{P(\text{blue})}$	$\frac{P(\text{dog})P(\text{green}\vert\text{dog})}{P(\text{green})}$	$\frac{P(\text{dog})P(\text{red}\vert\text{dog})}{P(\text{red})}$	
Column Total	$P(\text{blue}) =$ $P(\text{cat})P(\text{blue}\vert\text{cat})+$ $P(\text{dog})P(\text{blue}\vert\text{dog})$	$P(\text{green}) =$ $P(\text{cat})P(\text{green}\vert\text{cat})+$ $P(\text{dog})P(\text{green}\vert\text{dog})$	$P(\text{red}) =$ $P(\text{cat})P(\text{red}\vert\text{cat})+$ $P(\text{dog})P(\text{red}\vert\text{dog})$	1

pet	blue	green	red	Row Total
cat Cell prob Row prob Col prob	 $P(\text{cat})P(\text{blue}\|\text{cat})$ $P(\text{blue}\|\text{cat})$ $\frac{P(\text{cat})P(\text{blue}\|\text{cat})}{P(\text{blue})}$	 $P(\text{cat})P(\text{green}\|\text{cat})$ $P(\text{blue}\|\text{cat})$ $\frac{P(\text{cat})P(\text{green}\|\text{cat})}{P(\text{green})}$	 $P(\text{cat})P(\text{red}\|\text{cat})$ $P(\text{red}\|\text{cat})$ $\frac{P(\text{cat})P(\text{red}\|\text{cat})}{P(\text{red})}$	 $P(\text{cat})$
dog Cell prob Row prob Col prob	 $P(\text{dog})P(\text{blue}\|\text{dog})$ $P(\text{blue}\|\text{dog})$ $\frac{P(\text{dog})P(\text{blue}\|\text{dog})}{P(\text{blue})}$	 $P(\text{dog})P(\text{green}\|\text{dog})$ $P(\text{green}\|\text{dog})$ $\frac{P(\text{dog})P(\text{green}\|\text{dog})}{P(\text{green})}$	 $P(\text{dog})P(\text{red}\|\text{dog})$ $P(\text{red}\|\text{dog})$ $\frac{P(\text{dog})P(\text{red}\|\text{dog})}{P(\text{red})}$	 $P(\text{dog})$
Column Total	$P(\text{blue}) =$ $P(\text{cat})P(\text{blue}\|\text{cat})+$ $P(\text{dog})P(\text{blue}\|\text{dog})$	$P(\text{green}) =$ $P(\text{cat})P(\text{green}\|\text{cat})+$ $P(\text{dog})P(\text{green}\|\text{dog})$	$P(\text{red}) =$ $P(\text{cat})P(\text{red}\|\text{cat})+$ $P(\text{dog})P(\text{red}\|\text{dog})$	1

pet	blue	
cat		
Cell prob	$P(\text{cat})P(\text{blue}\mid\text{cat})$	$= P(\text{cat \& blue})$
Row prob	$P(\text{blue}\mid\text{cat})$	
Col prob	$\frac{P(\text{cat})P(\text{blue}\mid\text{cat})}{P(\text{blue})}$	$= P(\text{cat}\mid\text{blue})$

Probability multiplication

Bayes Rule

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat})P(\text{blue}\|\text{cat})$	$P(\text{cat})P(\text{green}\|\text{cat})$	$P(\text{cat})P(\text{red}\|\text{cat})$	$P(\text{cat})$
Row prob	$P(\text{blue}\|\text{cat})$	$P(\text{blue}\|\text{cat})$	$P(\text{red}\|\text{cat})$	
Col prob	$\frac{P(\text{cat})P(\text{blue}\|\text{cat})}{P(\text{blue})}$	$\frac{P(\text{cat})P(\text{green}\|\text{cat})}{P(\text{green})}$	$\frac{P(\text{cat})P(\text{red}\|\text{cat})}{P(\text{red})}$	
dog				
Cell prob	$P(\text{dog})P(\text{blue}\|\text{dog})$	$P(\text{dog})P(\text{green}\|\text{dog})$	$P(\text{dog})P(\text{red}\|\text{dog})$	$P(\text{dog})$
Row prob	$P(\text{blue}\|\text{dog})$	$P(\text{green}\|\text{dog})$	$P(\text{red}\|\text{dog})$	
Col prob	$\frac{P(\text{dog})P(\text{blue}\|\text{dog})}{P(\text{blue})}$	$\frac{P(\text{dog})P(\text{green}\|\text{dog})}{P(\text{green})}$	$\frac{P(\text{dog})P(\text{red}\|\text{dog}}{P(\text{red})}$	
Column Total	$P(\text{blue}) =$ $P(\text{cat})P(\text{blue}\|\text{cat})+$ $P(\text{dog})P(\text{blue}\|\text{dog})$	$P(\text{green}) =$ $P(\text{cat})P(\text{green}\|\text{cat})+$ $P(\text{dog})P(\text{green}\|\text{dog})$	$P(\text{red}) =$ $P(\text{cat})P(\text{red}\|\text{cat})+$ $P(\text{dog})P(\text{red}\|\text{dog})$	1

	B_1	B_2	B_3	Row Total
A_1				
Cell prob	$P(A_1)P(B_1 \vert A_1)$	$P(A_1)P(B_2 \vert A_1)$	$P(A_1)P(B_3 \vert A_1)$	$P(A_1)$
Row prob	$P(B_1 \vert A_1)$	$P(B_1 \vert A_1)$	$P(B_3 \vert A_1)$	
Col prob	$\frac{P(A_1)P(B_1 \vert A_1)}{P(B_1)}$	$\frac{P(A_1)P(B_2 \vert A_1)}{P(B_2)}$	$\frac{P(A_1)P(B_3 \vert A_1)}{P(B_3)}$	
A_2				
Cell prob	$P(A_2)P(B_1 \vert A_2)$	$P(A_2)P(B_2 \vert A_2)$	$P(A_2)P(B_3 \vert A_2)$	$P(A_2)$
Row prob	$P(B_1 \vert A_2)$	$P(B_2 \vert A_2)$	$P(B_3 \vert A_2)$	
Col prob	$\frac{P(A_2)P(B_1 \vert A_2)}{P(B_1)}$	$\frac{P(A_2)P(B_2 \vert A_2)}{P(B_2)}$	$\frac{P(A_2)P(B_3 \vert A_2)}{P(B_3)}$	
Column Total	$P(B_1) =$ $P(A_1)P(B_1 \vert A_1)+$ $P(A_2)P(B_1 \vert A_2)$	$P(B_2) =$ $P(A_1)P(B_2 \vert A_1)+$ $P(A_2)P(B_2 \vert A_2)$	$P(B_3) =$ $P(A_1)P(B_3 \vert A_1)+$ $P(A_2)P(B_3 \vert A_2)$	1

	B_1	B_2	B_3	$\cdots$	B_c	Row Total
A_1						
Cell prob	$P(A_1)P(B_1 \mid A_1)$	$P(A_1)P(B_2 \mid A_1)$	$P(A_1)P(B_3 \mid A_1)$	$\cdots$	$P(A_1)P(B_c \mid A_1)$	$P(A_1)$
Row prob	$P(B_1 \mid A_1)$	$P(B_1 \mid A_1)$	$P(B_3 \mid A_1)$	$\cdots$	$P(B_c \mid A_1)$	
Col prob	$\frac{P(A_1)P(B_1 \mid A_1)}{P(B_1)}$	$\frac{P(A_1)P(B_2 \mid A_1)}{P(B_2)}$	$\frac{P(A_1)P(B_3 \mid A_1)}{P(B_3)}$	$\cdots$	$\frac{P(A_1)P(B_c \mid A_1)}{P(B_c)}$	
A_2						
Cell prob	$P(A_2)P(B_1 \mid A_2)$	$P(A_2)P(B_2 \mid A_2)$	$P(A_2)P(B_3 \mid A_2)$	$\cdots$	$P(A_2)P(B_c \mid A_2)$	$P(A_2)$
Row prob	$P(B_1 \mid A_2)$	$P(B_2 \mid A_2)$	$P(B_3 \mid A_2)$	$\cdots$	$P(B_c \mid A_2)$	
Col prob	$\frac{P(A_2)P(B_1 \mid A_2)}{P(B_1)}$	$\frac{P(A_2)P(B_2 \mid A_2)}{P(B_2)}$	$\frac{P(A_2)P(B_3 \mid A_2)}{P(B_3)}$	$\cdots$	$\frac{P(A_2)P(B_c \mid A_2)}{P(B_c)}$	
Column Total	$P(B_1) =$ $P(A_1)P(B_1 \mid A_1)+$ $P(A_2)P(B_1 \mid A_2)$	$P(B_2) =$ $P(A_1)P(B_2 \mid A_1)+$ $P(A_2)P(B_2 \mid A_2)$	$P(B_3) =$ $P(A_1)P(B_3 \mid A_1)+$ $P(A_2)P(B_3 \mid A_2)$	$\cdots$	$P(B_c) =$ $P(A_1)P(B_c \mid A_1)+$ $P(A_2)P(B_c \mid A_2)$	1

	B_1	B_2	B_3	$\cdots$	B_c	Row Total
A_1						
Cell prob	$P(A_1)P(B_1\|A_1)$	$P(A_1)P(B_2\|A_1)$	$P(A_1)P(B_3\|A_1)$	$\cdots$	$P(A_1)P(B_c\|A_1)$	$P(A_1)$
Row prob	$P(B_1\|A_1)$	$P(B_1\|A_1)$	$P(B_3\|A_1)$	$\cdots$	$P(B_c\|A_1)$	
Col prob	$\frac{P(A_1)P(B_1\|A_1)}{P(B_1)}$	$\frac{P(A_1)P(B_2\|A_1)}{P(B_2)}$	$\frac{P(A_1)P(B_3\|A_1)}{P(B_3)}$	$\cdots$	$\frac{P(A_1)P(B_c\|A_1)}{P(B_c)}$	
A_2						
Cell prob	$P(A_2)P(B_1\|A_2)$	$P(A_2)P(B_2\|A_2)$	$P(A_2)P(B_3\|A_2)$	$\cdots$	$P(A_2)P(B_c\|A_2)$	$P(A_2)$
Row prob	$P(B_1\|A_2)$	$P(B_2\|A_2)$	$P(B_3\|A_2)$	$\cdots$	$P(B_c\|A_2)$	
Col prob	$\frac{P(A_2)P(B_1\|A_2)}{P(B_1)}$	$\frac{P(A_2)P(B_2\|A_2)}{P(B_2)}$	$\frac{P(A_2)P(B_3\|A_2)}{P(B_3)}$	$\cdots$	$\frac{P(A_2)P(B_c\|A_2)}{P(B_c)}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
A_r						
Cell prob	$P(A_r)P(B_1\|A_r)$	$P(A_r)P(B_2\|A_r)$	$P(A_r)P(B_3\|A_r)$	$\cdots$	$P(A_r)P(B_c\|A_r)$	$P(A_r)$
Row prob	$P(B_1\|A_r)$	$P(B_2\|A_r)$	$P(B_3\|A_r)$	$\cdots$	$P(B_c\|A_r)$	
Col prob	$\frac{P(A_r)P(B_1\|A_r)}{P(B_1)}$	$\frac{P(A_r)P(B_2\|A_r)}{P(B_2)}$	$\frac{P(A_r)P(B_3\|A_r)}{P(B_3)}$	$\cdots$	$\frac{P(A_r)P(B_c\|A_r)}{P(B_c)}$	
Column Total	$P(B_1) =$ $P(A_1)P(B_1\|A_1)+$ $P(A_2)P(B_1\|A_2)+$ $\cdots$ $P(A_r)P(B_1\|A_r)$	$P(B_2) =$ $P(A_1)P(B_2\|A_1)+$ $P(A_2)P(B_2\|A_2)+$ $\cdots$ $P(A_r)P(B_2\|A_r)$	$P(B_3) =$ $P(A_1)P(B_3\|A_1)+$ $P(A_2)P(B_3\|A_2)+$ $\cdots$ $P(A_r)P(B_3\|A_r)$	$\cdots$	$P(B_c) =$ $P(A_1)P(B_c\|A_1)+$ $P(A_2)P(B_c\|A_2)+$ $\cdots$ $P(A_r)P(B_c\|A_r)$	1

	B_1	B_2	B_3	$\cdots$	B_c	Row
A_1						
Cell prob	$P(A_1)P(B_1 \mid A_1)$	$P(A_1)P(B_2 \mid A_1)$	$P(A_1)P(B_3 \mid A_1)$	$\cdots$	$P(A_1)P(B_c \mid A_1)$	$P(A_1)$
Row prob	$P(B_1 \mid A_1)$	$P(B_1 \mid A_1)$	$P(B_3 \mid A_1)$	$\cdots$	$P(B_c \mid A_1)$	
Col prob	$\frac{P(A_1)P(B_1 \mid A_1)}{P(B_1)}$	$\frac{P(A_1)P(B_2 \mid A_1)}{P(B_2)}$	$\frac{P(A_1)P(B_3 \mid A_1)}{P(B_3)}$	$\cdots$	$\frac{P(A_1)P(B_c \mid A_1)}{P(B_c)}$	
A_2						
Cell prob	$P(A_2)P(B_1 \mid A_2)$	$P(A_2)P(B_2 \mid A_2)$	$P(A_2)P(B_3 \mid A_2)$	$\cdots$	$P(A_2)P(B_c \mid A_2)$	$P(A_2)$
Row prob	$P(B_1 \mid A_2)$	$P(B_2 \mid A_2)$	$P(B_3 \mid A_2)$	$\cdots$	$P(B_c \mid A_2)$	
Col prob	$\frac{P(A_2)P(B_1 \mid A_2)}{P(B_1)}$	$\frac{P(A_2)P(B_2 \mid A_2)}{P(B_2)}$	$\frac{P(A_2)P(B_3 \mid A_2)}{P(B_3)}$	$\cdots$	$\frac{P(A_2)P(B_c \mid A_2)}{P(B_c)}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
A_r						
Cell prob	$P(A_r)P(B_1 \mid A_r)$	$P(A_r)P(B_2 \mid A_r)$	$P(A_r)P(B_3 \mid A_r)$	$\cdots$	$P(A_r)P(B_c \mid A_r)$	$P(A_r)$
Row prob	$P(B_1 \mid A_r)$	$P(B_2 \mid A_r)$	$P(B_3 \mid A_r)$	$\cdots$	$P(B_c \mid A_r)$	
Col prob	$\frac{P(A_r)P(B_1 \mid A_r)}{P(B_1)}$	$\frac{P(A_r)P(B_2 \mid A_r)}{P(B_2)}$	$\frac{P(A_r)P(B_3 \mid A_r)}{P(B_3)}$	$\cdots$	$\frac{P(A_r)P(B_c \mid A_r)}{P(B_c)}$	
Column Total	$P(B_1) = \sum_{i=1}^{r} P(A_i)P(B_1 \mid A_i)$	$P(B_2) = \sum_{i=1}^{r} P(A_i)P(B_2 \mid A_i)$	$P(B_3) = \sum_{i=1}^{r} P(A_i)P(B_3 \mid A_i)$	$\cdots$	$P(B_c) = \sum_{i=1}^{r} P(A_i)P(B_c \mid A_i)$	1

	B_1
A_1	
Cell prob	$P(A_1)P(B_1\|A_1)$
Row prob	$P(B_1\|A_1)$
Col prob	$\frac{P(A_1)P(B_1\|A_1)}{P(B_1)}$
Column Total	$P(B_1) = \sum_{i=1}^{r} P(A_i)P(B_1\|A_i)$

Bayes Rule: $$P(A_1|B_1) = \frac{P(A_1)P(B_1|A_1)}{P(B_1)} = \frac{P(A_1)P(B_1|A_1)}{\sum_{i=1}^{r} P(A_i)P(B_1|A_i)}$$

	B_1
A_1	
Cell prob	$P(A_1)P(B_1\|A_1)$
Row prob	$P(B_1\|A_1)$
Col prob	$\frac{P(A_1)P(B_1\|A_1)}{P(B_1)}$
Column Total	$P(B_1) = \sum_{i=1}^{r} P(A_i)P(B_1\|A_i)$

Law of total probability: $$P(B_1) = \sum_{i=1}^{r} P(A_i)P(B_1|A_i)$$

	B_1
A_1	
Cell prob	$P(A_1)P(B_1 \mid A_1)$
Row prob	$P(B_1 \mid A_1)$
Col prob	$\frac{P(A_1)P(B_1 \mid A_1)}{P(B_1)}$
Column Total	$P(B_1) = \sum_{i=1}^{r} P(A_i)P(B_1 \mid A_i)$

Law of total probability:

$$P(B_1) = \sum_{i=1}^{r} P(A_i)P(B_1|A_i)$$

OR

$$P(B_1) = \sum_{i=1}^{r} P(A_i \;\&\; B_1)$$

Where are the OR probabilities?

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat \& blue})$	$P(\text{cat \& green})$	$P(\text{cat \& red})$	$P(\text{cat})$
Row prob	$P(\text{blue} \mid \text{cat})$	$P(\text{green} \mid \text{cat})$	$P(\text{red} \mid \text{cat})$	
Col prob	$P(\text{cat} \mid \text{blue})$	$P(\text{cat} \mid \text{green})$	$P(\text{cat} \mid \text{red})$	
dog				
Cell prob	$P(\text{dog \& blue})$	$P(\text{dog \& green})$	$P(\text{dog \& red})$	$P(\text{dog})$
Row prob	$P(\text{blue} \mid \text{dog})$	$P(\text{green} \mid \text{dog})$	$P(\text{red} \mid \text{dog})$	
Col prob	$P(\text{dog} \mid \text{blue})$	$P(\text{dog} \mid \text{green})$	$P(\text{dog} \mid \text{red})$	
Column Total	P(blue)	P(green)	P(red)	1

Where are the OR probabilities?

$$P(\text{cat or green}) = ?$$

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat \& blue})$	$P(\text{cat \& green})$	$P(\text{cat \& red})$	$P(\text{cat})$
Row prob	$P(\text{blue} \mid \text{cat})$	$P(\text{green} \mid \text{cat})$	$P(\text{red} \mid \text{cat})$	
Col prob	$P(\text{cat} \mid \text{blue})$	$P(\text{cat} \mid \text{green})$	$P(\text{cat} \mid \text{red})$	
dog				
Cell prob	$P(\text{dog \& blue})$	$P(\text{dog \& green})$	$P(\text{dog \& red})$	$P(\text{dog})$
Row prob	$P(\text{blue} \mid \text{dog})$	$P(\text{green} \mid \text{dog})$	$P(\text{red} \mid \text{dog})$	
Col prob	$P(\text{dog} \mid \text{blue})$	$P(\text{dog} \mid \text{green})$	$P(\text{dog} \mid \text{red})$	
Column Total	P(blue)	P(green)	P(red)	1

Where are the OR probabilities?

$$P(\text{cat or green}) = ?$$

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat \& blue})$	$P(\text{cat \& green})$	$P(\text{cat \& red})$	$P(\text{cat})$
Row prob	$P(\text{blue} \mid \text{cat})$	$P(\text{green} \mid \text{cat})$	$P(\text{red} \mid \text{cat})$	
Col prob	$P(\text{cat} \mid \text{blue})$	$P(\text{cat} \mid \text{green})$	$P(\text{cat} \mid \text{red})$	
dog				
Cell prob	$P(\text{dog \& blue})$	$P(\text{dog \& green})$	$P(\text{dog \& red})$	$P(\text{dog})$
Row prob	$P(\text{blue} \mid \text{dog})$	$P(\text{green} \mid \text{dog})$	$P(\text{red} \mid \text{dog})$	
Col prob	$P(\text{dog} \mid \text{blue})$	$P(\text{dog} \mid \text{green})$	$P(\text{dog} \mid \text{red})$	
Column Total	P(blue)	P(green)	P(red)	1

Where are the OR probabilities?

$$P(\text{cat or green}) =$$
$$P(\text{cat \& blue}) + P(\text{cat \& green}) +$$
$$P(\text{cat \& red}) + P(\text{dog \& green})$$

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat \& blue})$	$P(\text{cat \& green})$	$P(\text{cat \& red})$	$P(\text{cat})$
Row prob	$P(\text{blue} \mid \text{cat})$	$P(\text{green} \mid \text{cat})$	$P(\text{red} \mid \text{cat})$	
Col prob	$P(\text{cat} \mid \text{blue})$	$P(\text{cat} \mid \text{green})$	$P(\text{cat} \mid \text{red})$	
dog				
Cell prob	$P(\text{dog \& blue})$	$P(\text{dog \& green})$	$P(\text{dog \& red})$	$P(\text{dog})$
Row prob	$P(\text{blue} \mid \text{dog})$	$P(\text{green} \mid \text{dog})$	$P(\text{red} \mid \text{dog})$	
Col prob	$P(\text{dog} \mid \text{blue})$	$P(\text{dog} \mid \text{green})$	$P(\text{dog} \mid \text{red})$	
Column Total	P(blue)	P(green)	P(red)	1

Where are the OR probabilities?

$$P(\text{cat or green}) = \boxed{P(\text{cat})} + \boxed{P(\text{green})} - P(\text{cat \& green})$$

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat \& blue})$	$P(\text{cat \& green})$	$P(\text{cat \& red})$	$P(\text{cat})$
Row prob	$P(\text{blue} \mid \text{cat})$	$P(\text{green} \mid \text{cat})$	$P(\text{red} \mid \text{cat})$	
Col prob	$P(\text{cat} \mid \text{blue})$	$P(\text{cat} \mid \text{green})$	$P(\text{cat} \mid \text{red})$	
dog				
Cell prob	$P(\text{dog \& blue})$	$P(\text{dog \& green})$	$P(\text{dog \& red})$	$P(\text{dog})$
Row prob	$P(\text{blue} \mid \text{dog})$	$P(\text{green} \mid \text{dog})$	$P(\text{red} \mid \text{dog})$	
Col prob	$P(\text{dog} \mid \text{blue})$	$P(\text{dog} \mid \text{green})$	$P(\text{dog} \mid \text{red})$	
Column Total	P(blue)	P(green)	P(red)	1

NOT outcome probabilities

$$P(\text{NOT blue}) = ?$$

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat \& blue})$	$P(\text{cat \& green})$	$P(\text{cat \& red})$	$P(\text{cat})$
Row prob	$P(\text{blue} \mid \text{cat})$	$P(\text{green} \mid \text{cat})$	$P(\text{red} \mid \text{cat})$	
Col prob	$P(\text{cat} \mid \text{blue})$	$P(\text{cat} \mid \text{green})$	$P(\text{cat} \mid \text{red})$	
dog				
Cell prob	$P(\text{dog \& blue})$	$P(\text{dog \& green})$	$P(\text{dog \& red})$	$P(\text{dog})$
Row prob	$P(\text{blue} \mid \text{dog})$	$P(\text{green} \mid \text{dog})$	$P(\text{red} \mid \text{dog})$	
Col prob	$P(\text{dog} \mid \text{blue})$	$P(\text{dog} \mid \text{green})$	$P(\text{dog} \mid \text{red})$	
Column Total	P(blue)	P(green)	P(red)	1

NOT outcome probabilities

$$P(\text{NOT blue}) = P(\text{green}) + P(\text{red})$$

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat} \mathbin{\&} \text{blue})$	$P(\text{cat} \mathbin{\&} \text{green})$	$P(\text{cat} \mathbin{\&} \text{red})$	$P(\text{cat})$
Row prob	$P(\text{blue} \mid \text{cat})$	$P(\text{green} \mid \text{cat})$	$P(\text{red} \mid \text{cat})$	
Col prob	$P(\text{cat} \mid \text{blue})$	$P(\text{cat} \mid \text{green})$	$P(\text{cat} \mid \text{red})$	
dog				
Cell prob	$P(\text{dog} \mathbin{\&} \text{blue})$	$P(\text{dog} \mathbin{\&} \text{green})$	$P(\text{dog} \mathbin{\&} \text{red})$	$P(\text{dog})$
Row prob	$P(\text{blue} \mid \text{dog})$	$P(\text{green} \mid \text{dog})$	$P(\text{red} \mid \text{dog})$	
Col prob	$P(\text{dog} \mid \text{blue})$	$P(\text{dog} \mid \text{green})$	$P(\text{dog} \mid \text{red})$	
Column Total	P(blue)	P(green)	P(red)	1

NOT outcome probabilities

$$P(\text{NOT blue}) = 1 - P(\text{blue})$$

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat \& blue})$	$P(\text{cat \& green})$	$P(\text{cat \& red})$	$P(\text{cat})$
Row prob	$P(\text{blue} \mid \text{cat})$	$P(\text{green} \mid \text{cat})$	$P(\text{red} \mid \text{cat})$	
Col prob	$P(\text{cat} \mid \text{blue})$	$P(\text{cat} \mid \text{green})$	$P(\text{cat} \mid \text{red})$	
dog				
Cell prob	$P(\text{dog \& blue})$	$P(\text{dog \& green})$	$P(\text{dog \& red})$	$P(\text{dog})$
Row prob	$P(\text{blue} \mid \text{dog})$	$P(\text{green} \mid \text{dog})$	$P(\text{red} \mid \text{dog})$	
Col prob	$P(\text{dog} \mid \text{blue})$	$P(\text{dog} \mid \text{green})$	$P(\text{dog} \mid \text{red})$	
Column Total	P(blue)	P(green)	P(red)	1

What if the conditional probabilities contained no information?

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat} \ \& \ \text{blue})$	$P(\text{cat} \ \& \ \text{green})$	$P(\text{cat} \ \& \ \text{red})$	$P(\text{cat})$
Row prob	$P(\text{blue} \mid \text{cat})$	$P(\text{green} \mid \text{cat})$	$P(\text{red} \mid \text{cat})$	
Col prob	$P(\text{cat} \mid \text{blue})$	$P(\text{cat} \mid \text{green})$	$P(\text{cat} \mid \text{red})$	
dog				
Cell prob	$P(\text{dog} \ \& \ \text{blue})$	$P(\text{dog} \ \& \ \text{green})$	$P(\text{dog} \ \& \ \text{red})$	$P(\text{dog})$
Row prob	$P(\text{blue} \mid \text{dog})$	$P(\text{green} \mid \text{dog})$	$P(\text{red} \mid \text{dog})$	
Col prob	$P(\text{dog} \mid \text{blue})$	$P(\text{dog} \mid \text{green})$	$P(\text{dog} \mid \text{red})$	
Column Total	P(blue)	P(green)	P(red)	1

What if the conditional probabilities contained no information?

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat \& blue}) = ?$	$P(\text{cat \& green}) = ?$	$P(\text{cat \& red}) = ?$	$P(\text{cat})$
Row prob	$P(\text{blue} \mid \text{cat}) = P(\text{blue})$	$P(\text{green} \mid \text{cat}) = P(\text{green})$	$P(\text{red} \mid \text{cat}) = P(\text{red})$	
Col prob	$P(\text{cat} \mid \text{blue}) = P(\text{cat})$	$P(\text{cat} \mid \text{green}) = P(\text{cat})$	$P(\text{cat} \mid \text{red}) = P(\text{cat})$	
dog				
Cell prob	$P(\text{dog \& blue}) = ?$	$P(\text{dog \& green}) = ?$	$P(\text{dog \& red}) = ?$	$P(\text{dog})$
Row prob	$P(\text{blue} \mid \text{dog}) = P(\text{blue})$	$P(\text{green} \mid \text{dog}) = P(\text{green})$	$P(\text{red} \mid \text{dog}) = P(\text{red})$	
Col prob	$P(\text{dog} \mid \text{blue}) = P(\text{dog})$	$P(\text{dog} \mid \text{green}) = P(\text{dog})$	$P(\text{dog} \mid \text{red}) = P(\text{dog})$	
Column Total	P(blue)	P(green)	P(red)	1

What if the conditional probabilities contained no information?

pet	blue	green	red
cat			
Cell prob	$P(\text{cat \& blue}) = P(\text{cat})P(\text{blue})$	$P(\text{cat \& green}) = P(\text{cat})P(\text{green})$	$P(\text{cat \& red}) = P(\text{cat})P(\text{r}$
Row prob	$P(\text{blue} \mid \text{cat}) = P(\text{blue})$	$P(\text{green} \mid \text{cat}) = P(\text{green})$	$P(\text{red} \mid \text{cat}) = P(\text{red})$
Col prob	$P(\text{cat} \mid \text{blue}) = P(\text{cat})$	$P(\text{cat} \mid \text{green}) = P(\text{cat})$	$P(\text{cat} \mid \text{red}) = P(\text{cat})$
dog			
Cell prob	$P(\text{dog \& blue}) = P(\text{dog})P(\text{blue})$	$P(\text{dog \& green}) = P(\text{dog})P(\text{green})$	$P(\text{dog \& red}) = P(\text{dog})P(\text{r}$
Row prob	$P(\text{blue} \mid \text{dog}) = P(\text{blue})$	$P(\text{green} \mid \text{dog}) = P(\text{green})$	$P(\text{red} \mid \text{dog}) = P(\text{red})$
Col prob	$P(\text{dog} \mid \text{blue}) = P(\text{dog})$	$P(\text{dog} \mid \text{green}) = P(\text{dog})$	$P(\text{dog} \mid \text{red}) = P(\text{dog})$
Column Total	P(blue)	P(green)	P(red)

What if the conditional probabilities contained no information?

This is independence

pet	blue	green	red	Row Total
cat				
Cell prob	$P(\text{cat})P(\text{blue})$	$P(\text{cat})P(\text{green})$	$P(\text{cat})P(\text{red})$	$P(\text{cat})$
Row prob	$P(\text{blue})$	$P(\text{green})$	$P(\text{red})$	
Col prob	$P(\text{cat})$	$P(\text{cat})$	$P(\text{cat})$	
dog				
Cell prob	$P(\text{dog})P(\text{blue})$	$P(\text{dog})P(\text{green})$	$P(\text{dog})P(\text{red})$	$P(\text{dog})$
Row prob	$P(\text{blue})$	$P(\text{green})$	$P(\text{red})$	
Col prob	$P(\text{dog})$	$P(\text{dog})$	$P(\text{dog})$	
Column Total	P(blue)	P(green)	P(red)	1

What if the conditional probabilities contained no information?

This is independence

pet		blue	green	red	Row Total
cat					$P(\text{cat})$
	Cell prob	$P(\text{cat})P(\text{blue})$	$P(\text{cat})P(\text{green})$	$P(\text{cat})P(\text{red})$	
	Row prob	$P(\text{blue})$	$P(\text{green})$	$P(\text{red})$	
	Col prob	$P(\text{cat})$	$P(\text{cat})$	$P(\text{cat})$	
dog					$P(\text{dog})$
	Cell prob	$P(\text{dog})P(\text{blue})$	$P(\text{dog})P(\text{green})$	$P(\text{dog})P(\text{red})$	
	Row prob	$P(\text{blue})$	$P(\text{green})$	$P(\text{red})$	
	Col prob	$P(\text{dog})$	$P(\text{dog})$	$P(\text{dog})$	
Column Total		P(blue)	P(green)	P(red)	1

All of the information is in the margins!

What if the conditional probabilities contained no information?

This is independence

pet	blue	green	red	Row Total
cat				$P(\text{cat})$
dog				$P(\text{dog})$
Column Total	P(blue)	P(green)	P(red)	1

All of the information is in the margins!

Cell probabilities can be calculated from the marginal probabilities.

Why do we care about independence?

Conditional probabilities are at the heart of predictions.

Independence of variables A & B

↓

no point in making a prediction of A from B

Practice Problems

HINT: USE THE TABLE OF CELL, ROW, COLUMN, & MARGINAL PROBABILITIES.

1. Create an empty table
2. Fill in the information provided in the question
3. Identify the requested probability
4. Use the rules of probability to fill-in the gaps in the table to calculate the probability in question

Q:

If 44% of college students have access to Netflix, 35% have access to Hulu, and 20% have access to both, then what is the probability that a randomly selected student has either Hulu or Netflix?

Product	Apple OS	Windows OS	
Laptop	a	b	.80
Desktop	c	.15	d
	.60	e	f

Q:

Suppose the table of probabilities described the computer type and operating system choices for the Vanderbilt student population.

Calculate
- P(Apple OS | Laptop)
- P(Laptop | Apple OS)
- P(Laptop and Apple OS)

Q:

Is computer type and computer operating system independent in the population from the previous question?

Q:

Suppose three machines generate widgets with a defect rate of 0.1, 0.01, and 0.001, respectively. If the machines generate the same number of widgets, what is the probability that a randomly selected widget is defective.

Q:

Machines A, B, and C generate widgets with a defect rate of 0.1, 0.01, and 0.001, respectively. If machine A generates twice as many widgets as B, and machine B generates twice as many widgets as machine C, what is the probability that a randomly selected widget is defective.

Q:

Machines A, B, and C generate widgets with a defect rate of 0.1, 0.01, and 0.001, respectively. Machine A generates twice as many widgets as B, and machine B generates twice as many widgets as machine C. If a randomly selected widget is defective, what is the probability that the widget came from machine A?

Q:

Suppose there are 5 coins, 4 of which are fair and one with P(tails) = .25. A coin is randomly selected and flipped 3 times. Calculate the following:

- P(fair coin selected | flip sequence = TTT)
- P(2 heads in 3 flips | biased coin selected)